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ADDITIVE FUNCTIONAL INEQUALITIES IN GENERALIZED QUASI-BANACH SPACES

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ABSTRACT. In this paper, we investigate the Hyers-Ulam stability of the following function inequalities

$$\begin{aligned}\|af(x)+bf(y)+cf(z)\| &\le \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| \quad (0<|K|<|a+b+c|),\\ \|af(x)+bf(y)+Kf(z)\| &\le \left\|Kf\left(\frac{ax+by}{K}+z\right)\right\| \quad (0<K<|a+b+K|)\end{aligned}$$

in generalized quasi-Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1,.)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(.,.)$. Given $\epsilon>0$, does there exist a $\delta 0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x.y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\le\delta$$

for all $x,y\in E$, and for some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\|f(x)-T(x)\|\le\delta$$

for all $x\in E$. Moreover, if $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in E$, then T is $\mathbb{R}$-linear. In 1978, Th.M. Rassias [3] proved the following theorem.

Theorem 1.1. *Let $f:E\to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\|\le\epsilon(\|x\|^p+\|y\|^p) \tag{1.1}$$

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for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Th.M. Rassias. On the other hand, J.M. Rassias [5] generalized the Hyers-Ulam stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.2. ([6, 7]) *If it is assumed that there exist constants $\Theta \ge 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : E \to E'$ is a mapping from a norm space E into a Banach space E' such that the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \Theta\|x\|^{p_1}\|y\|^{p_2}$$

for all $x, y \in E$, then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \frac{\Theta}{2 - 2^p}\|x\|^p,$$

for all $x \in E$. If, in addition, $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear

More generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings can be found in [8]–[22].

In [23], Park et al. investigated the following inequalities

$$\|f(x) + f(y) + f(z)\| \le \left\|2f\left(\frac{x + y + z}{2}\right)\right\|,$$

$$\|f(x) + f(y) + f(z)\| \le \|f(x + y + z)\|,$$

$$\|f(x) + f(y) + 2f(z)\| \le \left\|2f\left(\frac{x + y}{2} + z\right)\right\|$$

in Banach spaces. Recently, Cho et al. [24] investigated the following functional inequality

$$\|f(x) + f(y) + f(z) \le \left\|Kf\left(\frac{x + y + z}{K}\right)\right\| \qquad (0 < |K| < |3|)$$

in non-Archimedean Banach spaces. Lu and Park [25] investigated the following functional inequality

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \le \left\|Kf\left(\frac{\sum_{i=1}^{N}(x_i)}{K}\right)\right\| \qquad (0 < |K| \le N)$$

in Fréchet spaces.

In [26], we investigated the following functional inequalities

$$\|f(x) + f(y) + f(z)\| \le \left\|Kf\left(\frac{x + y + z}{K}\right)\right\| \qquad (0 < |K| < 3), \tag{1.3}$$

$$\|f(x)+f(y)+Kf(z)\| \le \left\|Kf\left(\frac{x+y}{K}+z\right)\right\| \qquad (0<K\neq 2) \tag{1.4}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.3) and (1.4) in Banach spaces.

We consider the following functional inequalities

$$\|af(x)+bf(y)+cf(z)\| \le \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| \quad (0<|K|<|a+b+c|), \tag{1.5}$$

$$\|af(x)+bf(y)+Kf(z)\| \le \left\|Kf\left(\frac{ax+by}{K}+z\right)\right\| \quad (0<K<|a+b+K|), \tag{1.6}$$

where a, b, c are nonzero real numbers.

Now, we recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.3. ([27, 28]) Let X be a linear space. A quasi-norm is a real-valued function on X satisfying the following:

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\|=0$ if and only if $x=0$.
(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(3) There is a constant $C \ge 1$ such that $\|x+y\| \le C(\|x\|+\|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X.

A *quasi-Banach space* is a complete quasi-normed space.

Baak [29] generalized the concept of quasi-normed spaces.

Definition 1.4. ([29]) Let X be a linear space. A **generalized quasi-norm** is a real-valued function on X satisfying the following:

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\|=0$ if and only if $x=0$.
(2) $\|\lambda x\| = |\lambda|\cdot\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(3) There is a constant $C \ge 1$ such that $\|\sum_{j=1}^{\infty} x_j\| \le \sum_{j=1}^{\infty} C\|x_j\|$ for all $x_1, x_2, \cdots \in X$ with $\sum_{j=1}^{\infty} x_j \in X$.

The pair $(X, \|\cdot\|)$ is called a *generalized quasi-normed space* if $\|\cdot\|$ is a generalized quasi-norm on X. The smallest possible C is called the *modulus of concavity* of $\|\cdot\|$.

A *generalized quasi-Banach space* is a complete generalized quasi-normed space.

In this paper, we show that the Hyers-Ulam stability of the functional inequalities (1.5) and (1.6) in generalized quasi-Banach spaces.

Throughout this paper, assume that X is a generalized quasi-normed vector space with generalized quasi-norm $\|\cdot\|$ and that $(Y, \|\cdot\|)$ is a generalized quasi-Banach space. Let C be the modulus of concavity of $\|\cdot\|$.

2. Hyers-Ulam stability of the functional inequality (1.5)

Throughout this section, assume that K is a real number with $0<|K|<|a+b+c|$.

Proposition 2.1. *Let $f: X \to Y$ be a mapping such that*

$$\|af(x)+bf(y)+cf(z)\| \le \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| \tag{2.1}$$

for all $x, y, z \in X$. Then the mapping $f: X \to Y$ is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$\|(a+b+c)f(0)\| \leq \|Kf(0)\|.$$

So $f(0) = 0$.

Letting $z = 0$ and $y = -\frac{b}{a}x$ in (2.1), we get

$$\left\|af(x) + bf\left(-\frac{a}{b}x\right)\right\| \leq \|Kf(0)\| = 0$$

for all $x \in X$. So $f(x) = -\frac{b}{a}f(-\frac{a}{b}x)$ for all $x \in X$.

Replacing x by $-x$ and letting $y = 0$ and $z = \frac{a}{c}x$ in (2.1), we get

$$\left\|af(-x) + cf\left(\frac{a}{c}x\right)\right\| \leq \|Kf(0)\| = 0$$

for all $x \in X$. So $f(-x) = -\frac{c}{a}f(\frac{a}{c}x)$ for all $x \in X$. Then we get

$$\begin{aligned}
\|f(x) + f(-x)\| &= \left\|-\frac{b}{a}f\left(-\frac{a}{b}x\right) - \frac{c}{a}f\left(\frac{a}{c}x\right)\right\| \\
&= \frac{1}{|a|}\left\|af(0) + bf\left(-\frac{a}{b}x\right) + cf\left(\frac{a}{c}x\right)\right\| \\
&\leq \frac{1}{|a|}\left\|Kf\left(\frac{a \cdot 0 - b\frac{a}{b}x + c\frac{a}{c}x}{K}\right)\right\| = 0
\end{aligned}$$

Thus $f(x) = -f(-x)$.

$$\begin{aligned}
\|f(x) + f(y) - f(x+y)\| &= \|f(x) + f(y) + f(-x-y)\| \\
&= \left\|-\frac{a}{a}f(-\frac{a}{a}x) - \frac{b}{a}f(-\frac{a}{b}y) - \frac{c}{a}f(\frac{ax+ay}{c})\right\| \\
&= \frac{1}{|a|}\left\|af(-\frac{a}{a}x) + bf(-\frac{a}{b}y) + cf(\frac{ax+ay}{c})\right\| \\
&= \frac{1}{|a|}\left\|Kf\left(\frac{a\cdot(-\frac{a}{a}x) + b\cdot(-\frac{a}{b}x) + c\cdot\frac{a(x+y)}{c}}{K}\right)\right\| = 0.
\end{aligned}$$

Thus

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$, as desired. □

Theorem 2.2. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|af(x) + bf(y) + cf(z)\| \leq \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| + \phi(x,y,z), \tag{2.2}$$

where $\phi : X^3 \to [0,\infty)$ satisfies $\phi(0,0,0) = 0$ and

$$\widetilde{\phi}(x,y,z) := \sum_{j=0}^{\infty}\left(\frac{c}{a}\right)^j \phi\left(\left(\frac{a}{c}\right)^j y, \left(\frac{a}{c}\right)^j z, \left(\frac{a}{c}\right)^j x\right) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|A(x) - f(x)\| \leq \frac{C^2}{|a|}\left[\widetilde{\phi}\left(x, -\frac{a}{b}x, 0\right) + \widetilde{\phi}\left(0, -\frac{a}{b}x, \frac{a}{c}x\right)\right] \tag{2.3}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.2), we get $\|(a+b+c)f(0)\| \le \|Kf(0)\| + \phi(0,0,0) = \|Kf(0)\|$. So $f(0) = 0$.

Letting $y = 0$ and $z = -\frac{a}{c}x$ in (2.2), we get

$$\left\|af(x) + cf\left(-\frac{a}{c}x\right)\right\| \le \phi\left(x, 0, -\frac{a}{c}x\right)$$

for all $x \in X$. So $\|f(x) + \frac{c}{a}f(-\frac{a}{c}x)\| \le \frac{1}{|a|}\phi\left(x, 0, -\frac{a}{c}x\right)$ for all $x \in X$.

Letting $y = -\frac{a}{b}x$ and $z = 0$ in (2.2), we obtain

$$\left\|f(x) + \frac{b}{a}f\left(-\frac{a}{b}x\right)\right\| \le \frac{1}{|a|}\phi\left(x, -\frac{a}{b}x, 0\right)$$

for all $x \in X$. So

$$\begin{aligned}\left\|f(x) - \frac{c}{a}f\left(\frac{a}{c}x\right)\right\| &= \left\|f(x) + \frac{b}{a}f\left(-\frac{ax}{b}\right) - \frac{b}{a}f\left(-\frac{ax}{b}\right) - \frac{c}{a}f\left(\frac{a}{c}x\right)\right\| \\ &\le C\left(\left\|f(x) + \frac{b}{a}f\left(-\frac{ax}{b}\right)\right\| + \left\|\frac{b}{a}f\left(-\frac{ax}{b}\right) + \frac{c}{a}f\left(\frac{a}{c}x\right)\right\|\right) \\ &\le \frac{C}{|a|}\left[\phi\left(x, -\frac{ax}{b}, 0\right) + \phi\left(0, -\frac{ax}{b}, \frac{ax}{c}\right)\right]\end{aligned} \tag{2.4}$$

for all $x \in X$.

It follows from (2.4) that

$$\begin{aligned}&\left\|(\frac{c}{a})^l f\left((\frac{a}{c})^l x\right) - (\frac{c}{a})^m f\left((\frac{a}{c})^m x\right)\right\| \\ &\le C\sum_{j=l}^{m-1}\left\|(\frac{c}{a})^j f\left((\frac{a}{c})^j x\right) - (\frac{c}{a})^{j+1} f\left((\frac{a}{c})^{j+1} x\right)\right\| \\ &\le \frac{C^2}{|a|}\sum_{j=l}^{m-1}\left(\frac{c}{a}\right)^j\left[\phi\left(\left(\frac{a}{c}\right)^j x, -\frac{a}{b}\left(\frac{a}{c}\right)^j x, 0\right) + \phi\left(0, -\frac{a}{b}\left(\frac{a}{c}\right)^j x, \left(\frac{a}{c}\right)^{j+1} x\right)\right]\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(\frac{c}{a})^n f((\frac{a}{c})^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(\frac{c}{a})^n f((\frac{a}{c})^n x)\}$ converges. We define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}\{(\frac{c}{a})^n f((\frac{a}{c})^n x)\}$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get (2.3).

Next, we show that $A : X \to Y$ is an additive mapping.

$$\begin{aligned}
\|A(x)+A(-x)\| &= \lim_{n\to\infty}(\frac{c}{a})^n \left\| f\left(\frac{a^n x}{c^n}\right) + f\left(\frac{-a^n x}{c^n}\right)\right\| \\
&\leq C\lim_{n\to\infty}(\frac{c}{a})^n \left[\left\| f\left(\frac{a^n x}{c^n}\right) + \frac{b}{a} f\left(-\frac{a}{b}\cdot\frac{a^n x}{c^n}\right)\right\|\right. \\
&+ \left\| f\left(-\frac{a^n x}{c^n}\right) + \frac{c}{a} f\left(\frac{a}{c}\cdot\frac{a^n x}{c^n}\right)\right\| \\
&+ \left.\left\| \frac{b}{a} f\left(-\frac{a}{b}\cdot\frac{a^n x}{c^n}\right) + \frac{c}{a} f\left(\frac{a}{c}\cdot\frac{a^n x}{c^n}\right)\right\|\right] \\
&\leq C\frac{1}{|a|}\lim_{n\to\infty}(\frac{c}{a})^n \left[\phi\left(\frac{a^n x}{c^n}, -\frac{a}{b}\frac{a^n x}{c^n}, 0\right) + \phi\left(-\frac{a^n x}{c^n}, 0, \frac{a^{n+1}x}{c^{n+1}}\right)\right. \\
&+ \left.\phi\left(0, -\frac{a}{b}\frac{a^n x}{c^n}, \frac{a^{n+1}x}{c^{n+1}}\right)\right] \\
&= 0
\end{aligned}$$

and so $A(-x) = -A(x)$ for all $x \in X$.

$$\begin{aligned}
&\|A(x)+A(y)-A(x+y)\|\| = \lim_{n\to\infty}(\frac{c}{a})^n \left\| f\left(\frac{a^n x}{c^n}\right) + f\left(\frac{a^n y}{c^n}\right) - f\left(\frac{a^n(x+y)}{c^n}\right)\right\| \\
&= C\lim_{n\to\infty}(\frac{c}{a})^n \left[\left\| f\left(\frac{a^n x}{c^n}\right) + \frac{b}{a} f\left(-\frac{a}{b}\frac{a^n x}{c^n}\right)\right\|\right. \\
&+ \left\| f\left(\frac{a^n y}{c^n}\right) + \frac{c}{a} f\left(-\frac{a^{n+1} y}{c^{n+1}}\right)\right\| \\
&+ \left.\left\| f\left(\frac{a^n(x+y)}{c^n}\right) + \frac{b}{a} f\left(-\frac{a}{b}\frac{a^n x}{c^n}\right) + \frac{c}{a} f\left(-\frac{a^{n+1} y}{c^{n+1}}\right)\right\|\right] \\
&\leq C\frac{1}{|a|}\lim_{n\to\infty}(\frac{c}{a})^n \left[\phi\left(\frac{a^n x}{c^n}, -\frac{a}{b}(\frac{a^n x}{c^n}), 0\right) + \phi\left(\frac{a^n y}{c^n}, 0, -\frac{a}{c}(\frac{a^n x}{c^n})\right)\right. \\
&+ \left.\phi\left(\frac{a^n(x+y)}{c^n}, -\frac{a}{b}(\frac{a^n x}{c^n}), -\frac{a}{c}(\frac{a^n x}{c^n})\right)\right] \\
&= 0
\end{aligned}$$

for all $x, y \in X$. Thus the mapping $A : X \to Y$ is additive.

Now, we prove the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (2.3). Then we obtain

$$\begin{aligned}
\|A(x)-T(x)\| &= (\frac{c}{a})^n \left\| A\left((\frac{a}{c})^n x\right) - T\left((\frac{a}{c})^n x\right)\right\| \\
&\leq C\cdot(\frac{c}{a})^n \left[\left\| A\left((\frac{a}{c})^n x\right) - f\left((\frac{a}{c})^n x\right)\right\|\right. \\
&+ \left.\left\| T\left((\frac{a}{c})^n x\right) - f\left((\frac{a}{c})^n x\right)\right\|\right] \\
&\leq 2C\frac{C^2}{|a|}\left[\widetilde{\phi}\left(x, -\frac{a}{b}x, 0\right) + \widetilde{\phi}(0, -\frac{a}{b}x, \frac{a}{c}x)\right]
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. Then we can conclude that $A(x) = T(x)$ for all $x \in X$. This complete the proof. □

Corollary 2.3. *Let p and θ be positive real numbers with $p > 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\|af(x) + bf(y) + cf(z)\| \leq \left\|Kf\left(\frac{ax + by + cz}{K}\right)\right\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{C}{|a|} \cdot \frac{c^p + a^p}{c^p - c(a+b)^{p-1}} \theta\|x\|^p$$

for all $x \in X$.

3. Hyers-Ulam stability of the functional inequality (1.6)

Throughout this section, assume that K, a, b are nonzero real numbers with $0 < K \neq 2$ and $|a + b + K| \geq K$.

Proposition 3.1. *Let $f : X \to Y$ be a mapping such that*

$$\|af(x) + bf(y) + Kf(z)\| \leq \left\|Kf\left(\frac{ax + by}{K} + z\right)\right\| \tag{3.1}$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is additive.

Proof. Letting $x = y = z = 0$ in (3.1), we get

$$\|(K + a + b)f(0)\| \leq \|Kf(0)\|.$$

So $f(0) = 0$.

Letting $y = -\frac{a}{b}x$ and $z = 0$ in (3.1), we get

$$\left\|af(x) + bf\left(-\frac{a}{b}x\right)\right\| \leq \|Kf(0)\| = 0$$

for all $x \in X$. So $f(x) = -\frac{b}{a}f(-\frac{a}{b}x)$ for all $x \in X$.

Replacing x by $-x$ and letting $y = 0$ and $z = \frac{a}{K}x$ in (3.1), we get

$$\left\|af(-x) + Kf\left(\frac{a}{K}x\right)\right\| \leq \|Kf(0)\| = 0$$

for all $x \in X$. So $f(-x) = -\frac{K}{a}f(\frac{a}{K}x)$ for all $x \in X$.

Thus we get

$$\|f(x) + f(-x)\| = \frac{1}{|a|}\left\|bf\left(-\frac{a}{b}x\right) + Kf\left(\frac{a}{K}x\right)\right\| \leq \frac{1}{|a|}\|f(0)\| = 0$$

for all $x \in X$. So $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = \frac{-x-y}{K}$ in (3.1), we get

$$\begin{aligned}\left\|af(x) + bf(y) - Kf\left(\frac{ax + by}{K}\right)\right\| &= \left\|af(x) + bf(y) + Kf\left(\frac{-ax - by}{K}\right)\right\| \\ &\leq \|Kf(0)\| = 0\end{aligned}$$

for all $x, y \in X$. Thus

$$Kf\left(\frac{ax + by}{K}\right) = af(x) + bf(y) \tag{3.2}$$

for all $x, y \in X$. Letting $y = 0$ in (3.2), we get $f(x) = \frac{a}{K} f\left(\frac{Kx}{a}\right)$ for all $x \in X$. Letting $x = 0$ in (3.2), we get $f(y) = \frac{b}{K} f\left(\frac{Ky}{b}\right)$. So

$$\|f(x) + f(y) - f(x+y)\| = \left\| \frac{a}{K} f\left(\frac{Kx}{a}\right) + \frac{b}{K} f\left(\frac{Ky}{b}\right) + f(-x-y) \right\|$$
$$= \frac{1}{|K|} \left\| af\left(\frac{Kx}{a}\right) + bf\left(\frac{Ky}{b}\right) + Kf(-x-y) \right\| = 0$$

for all $x, y \in X$, as desired. □

Theorem 3.2. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|af(x) + bf(y) + Kf(z)\| \leq \left\| Kf\left(\frac{ax+by}{K} + z\right) \right\| + \phi(x, y, z), \tag{3.3}$$

where $\phi : X^3 \to [0, \infty)$ satisfies $\phi(0, 0, 0) = 0$ and

$$\widetilde{\phi}(x, y, z) := \sum_{j=1}^{\infty} \left| \left(\frac{a}{K}\right)^j \right| \phi\left(\left(\frac{K}{a}\right)^j x, \left(\frac{K}{a}\right)^j y, \left(\frac{K}{a}\right)^j z \right) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|A(x) - f(x)\| \leq \frac{C^2}{|K|} \left[\widetilde{\phi}\left(0, -\frac{K}{a}x, x\right) + \widetilde{\phi}\left(\frac{K}{a}x, -\frac{K}{b}x, 0\right) \right] \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.3), we get $\|(K + a + b)f(0)\| \leq \|Kf(0)\| + \phi(0, 0, 0) = \|Kf(0)\|$. So $f(0) = 0$.

Letting $x = 0, y = -\frac{Kx}{b}, z = x$ in (3.3), we obtain

$$\left\| af(0) + bf\left(-\frac{K}{b}x\right) + Kf(x) \right\| \leq \phi\left(0, -\frac{K}{b}x, x\right)$$

for all $x \in X$.

Letting $y = 0, z = -\frac{Kx}{a}$in (3.3), we obtain

$$\left\| af(x) + bf(0) + Kf\left(\frac{-ax}{K}\right) \right\| \leq \phi\left(x, 0, -\frac{ax}{K}\right)$$

for all $x \in X$.

Letting $x = \frac{Kx}{a}, y = -\frac{Kx}{b}, z = 0$ in (3.3), we get

$$\left\| af\left(\frac{Kx}{a}\right) + bf\left(-\frac{Kx}{b}\right) + Kf(0) \right\| \leq \phi\left(\frac{Kx}{a}, -\frac{Kx}{b}, 0\right)$$

for all $x \in X$. So

$$\begin{aligned} &\left\| f(x) - \frac{a}{K} f\left(\frac{K}{a}x\right) \right\| \\ &\leq C\left[\left\| f(x) + \frac{b}{K} f\left(-\frac{Kx}{b}\right) \right\| + \left\| \frac{b}{K} f\left(-\frac{K}{b}x\right) + \frac{a}{K} f\left(\frac{K}{a}x\right) \right\| \right] \\ &\leq \frac{C}{|K|} \left[\phi\left(0, -\frac{K}{b}x, x\right) + \phi\left(\frac{K}{a}x, -\frac{K}{b}x, 0\right) \right] \end{aligned} \tag{3.5}$$

for all $x \in X$. It follows from (3.5) that

$$\begin{aligned}
&\left\|\left(\frac{a}{K}\right)^l f\left(\left(\frac{K}{a}\right)^l x\right) - \left(\frac{a}{K}\right)^m f\left(\left(\frac{K}{a}\right)^m x\right)\right\| \\
&\le C\sum_{j=l}^{m-1}\left\|\left(\frac{a}{K}\right)^j f\left(\left(\frac{K}{a}\right)^j x\right) - \left(\frac{a}{K}\right)^{j+1} f\left(\left(\frac{K}{a}\right)^{j+1} x\right)\right\| \\
&\le C^2\sum_{j=l}^{m-1}\left|\left(\frac{a}{K}\right)^j\right|\left[\left\|f(\left(\left(\frac{K}{a}\right)^j x\right) + \frac{b}{K}f\left(-\frac{K}{b}\left(\left(\frac{K}{a}\right)^j x\right)\right))\right\|\right. \\
&\left.+\left\|\frac{b}{K}f\left(-\frac{K}{b}\left(\left(\frac{K}{a}\right)^j x\right)\right) + \frac{a}{K}f\left(\frac{K}{a}\left(\left(\frac{K}{a}\right)^j x\right)\right)\right\|\right] \\
&\le \frac{C^2}{|K|}\sum_{j=l}^{m-1}\left|\left(\frac{a}{K}\right)^j\right|\left[\phi\left(0, -\frac{K}{a}\left(\frac{K}{a}\right)^j x, \left(\frac{K}{a}\right)^j x\right) + \phi\left(\frac{K}{a}\left(\frac{K}{a}\right)^j x, -\frac{K}{b}\left(\frac{K}{a}\right)^j x, 0\right)\right]
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(\frac{a}{K})^n f((\frac{K}{a})^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(\frac{a}{K})^n f((\frac{K}{a})^n x)\}$ converges. So we may define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}((\frac{a}{K})^n f((\frac{K}{a})^n x))$ for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get (3.4).

Now, we show that A is additive.

$$\begin{aligned}
&\|A(x) + A(y) - A(x+y)\| \\
&= \lim_{n\to\infty}\left|\frac{a}{K}\right|^n\left\|f((\frac{K}{a})^n x) + f((\frac{K}{a})^n y) - f((\frac{K}{a})^n(x+y))\right\| \\
&\le C\lim_{n\to\infty}\left|\frac{a}{K}\right|^n\left[\left\|f\left(\left(\frac{K}{a}\right)^n x\right) + \frac{b}{K}f\left(-\frac{K}{b}\left(\frac{K}{a}\right)^n x\right)\right\|\right. \\
&+\left\|f\left(\left(\frac{K}{a}\right)^n y\right) + \frac{a}{K}f\left(-\frac{K}{a}\left(\frac{K}{a}\right)^n y\right)\right\| \\
&\left.+\left\|\frac{a}{K}f\left(-\frac{K}{a}\left(\frac{K}{a}\right)^n y\right) + \frac{b}{K}f\left(-\frac{K}{b}\left(\frac{K}{a}\right)^n x\right) + f\left(\left(\frac{K}{a}\right)^n (x+y)\right)\right\|\right] \\
&\le C\lim_{n\to\infty}\left|\frac{a}{K}\right|^n\left[\phi\left(0, -\frac{K}{b}\left(\frac{K}{a}\right)^n x, \left(\frac{K}{a}\right)^n x\right)\right. \\
&+\phi\left(-\frac{K}{a}\left(\frac{K}{a}\right)^n y, 0, \left(\frac{K}{a}\right)^n y\right) \\
&\left.+\phi\left(-\frac{K}{a}\left(\frac{K}{a}\right)^n y, -\frac{K}{b}\left(\frac{K}{a}\right)^n x, \left(\frac{K}{a}\right)^n (x+y)\right)\right] \\
&= 0
\end{aligned}$$

for all $x, y \in X$. So the mapping $A : X \to Y$ is an additive mapping.

Now, we show that the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (3.4). Then we get

$$\begin{aligned}
&\|A(x) - T(x)\| = \lim_{n\to\infty} \left|\frac{a}{K}\right|^n \left\|A\left|\left(\frac{K}{a}\right|^n x\right) - T\left(\left(\frac{K}{a}\right)^n x\right)\right\| \\
&\leq C \lim_{n\to\infty} \left|\frac{a}{K}\right|^n \left[\left\|A\left(\left(\frac{K}{a}\right)^n x\right) - f\left(\left(\frac{K}{a}\right)^n x\right)\right\| + \left\|T\left(\left(\frac{K}{a}\right)^n x\right) - f\left(\left(\frac{K}{a}\right)^n x\right)\right\|\right] \\
&\leq 2C\frac{C^2}{|K|} \lim_{n\to\infty} \left[\tilde{\phi}\left(0, -\frac{K}{a}\left(\frac{K}{a}\right)^n x, \left(\frac{K}{a}\right)^n x\right)\right) + \tilde{\phi}\left(\frac{K}{a}\left(\frac{K}{a}\right)^n x, -\frac{K}{b}\left(\frac{K}{a}\right)^n x, 0\right)\right] \\
&= 0
\end{aligned}$$

for all $x \in X$. Thus we may conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. So the mapping $A : X \to Y$ is a unique additive mapping satisfying (3.4). □

Corollary 3.3. *Let p, θ and K be positive real numbers with $p > 1$ and $|a + b + K| > K$. Let $f : X \to Y$ be a mapping satisfying*

$$\|af(x) + bf(y) + Kf(z)\| \leq \left\|Kf\left(\frac{ax + by}{K} + z\right)\right\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{\frac{1}{K}\left(\frac{a}{K}\right)^p + \frac{3a}{K}}{\left(\frac{a}{K}\right)^p - \frac{a}{K}}\theta\|x\|^p$$

for all $x \in X$.

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